Natural Transformation Objects as Equalisers

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We will provide a definition of the natural transformation object in an enriched category courtesy of [1] and prove that the natural transformation object is given by an equaliser.

Definition 0.1. Let V be a symmetric monoidal category with symmetry τ and let $F, G\colon C \to D$ be V-enriched functors. A **natural transformation object** $\mathrm{NT}_V(F,G)$ is an object of V such that for each $a \in \mathrm{ob}(C)$ there is a map $N_a\colon \mathrm{NT}_V(F,G) \to D(F(a),G(a))$ such that the following diagram commutes.

$$\begin{array}{ccc}
\mathrm{NT}_V(F,G)\otimes C(a,b) & \xrightarrow{(G\otimes N_a)\circ\tau} & D(G(a),G(b))\otimes D(F(a),G(a)) \\
\Big\downarrow{\scriptstyle N_b\otimes F} & & \Big\downarrow{\scriptstyle\circ} \\
D(F(b),G(b))\otimes D(F(a),F(b)) & \xrightarrow{\circ} & D(F(a),G(b))
\end{array}$$

Additionally we require $NT_V(F,G)$ is unique up to isomorphism by the universal property which states that for any object $M \in V$ and collection of morphisms $\{M_a\colon M \to D(F(a),G(a))\}_{a\in C}$ there exists a unique map $\phi\colon M \to NT(F,G)$ such that $(\phi \otimes \mathrm{id}_{C(a,b)})$ factors through the above diagram.

Theorem 0.2. *If V a closed and complete symmetric monoidal category then for two V-enriched functors $F,G\colon C \to D$ the natural transformation object is given by an equaliser of the form below.*

$$\mathrm{NT}_V(F,G) \xrightarrow{\prod_c N_c} \prod_c D(F(c),G(c)) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \prod_{a,b}[C(a,b),D(F(a),G(b))]$$

To prove this theorem we need the help of the following intermediary lemma.

Lemma 0.3. *If V is a closed monoidal category then there are maps λ_1 and λ_2 which can be found such that the commuting of the diagram in Definition*

0.1 is equivalent to the commuting of the following diagram.

$$\begin{array}{ccc} \mathrm{NT}_V(F,G) & \xrightarrow{N_a} & D(F(a),G(a)) \\ \downarrow{\scriptstyle N_b} & & \downarrow{\scriptstyle \lambda_1} \\ D(F(b),G(b)) & \xrightarrow{\lambda_2} & [C(a,b), D(F(a),G(b))] \end{array}$$

Proof. Since V is a closed category then for each object $C \in V$ we have an adjunction $- \otimes C \dashv [C, -]$ and hence a unit natural transformation $\eta^C \colon \mathrm{id} \to [C, - \otimes C]$. For objects $NT, D \in V$ and a morphism $P \colon NT \otimes C \to D$ then we can take the composition with η^C_{NT} to get a map

$$\hat{P} \colon NT \xrightarrow{\eta^C_{NT}} [C, NT \otimes C] \xrightarrow{[\mathrm{id}_C, P]} [C, D]$$

If we have objects $D_N, D_K \in V$ and morphisms $N \colon NT \to D_N$, $F \colon C \to D_K$, and $\circ(\sigma) \colon D_N \otimes D_K \to D$ and write P as a composite of the form

$$NT \otimes C \xrightarrow{N \otimes K} D_N \otimes D_K \xrightarrow{\circ(\sigma)} D$$

then we can see that $\hat{P}$ is given by the composite

$$NT \xrightarrow{\eta^C_{NT}} [C, NT \otimes C] \xrightarrow{[\mathrm{id}_C, N \otimes K]} [C, D_N \otimes D_K] \xrightarrow{[\mathrm{id}_C, \circ(\sigma)]} [C, D].$$

We can write $N \otimes K$ as two maps $(\mathrm{id}_{D_N} \otimes K) \circ (N \otimes \mathrm{id}_C)$ and re-write $\hat{P}$ as a map

$$NT \xrightarrow{\eta^C_{NT}} [C, NT \otimes C] \xrightarrow{[id_C, N \otimes \mathrm{id}_C]} [C, D_N \otimes C] \xrightarrow{[\mathrm{id}_C, \mathrm{id}_{D_N} \otimes K]} [C, D_N \otimes D_K] \xrightarrow{[\mathrm{id}_C, \circ(\sigma)]} [C, D].$$

Now naturality means η^C commutes with morphisms. In particular

$$[\mathrm{id}_C, N \otimes \mathrm{id}_C] \circ \eta^C_{NT} = \eta^C_{D_N} \circ N.$$

We re-write $\hat{P}$ with the morphisms having commuted to get

$$\hat{P} \colon NT \xrightarrow{N} D_N \xrightarrow{\eta_{D_N}} [C, D_N \otimes C] \xrightarrow{[\mathrm{id}_C, \mathrm{id}_{D_N} \otimes K]} [C, D_N \otimes D_K] \xrightarrow{[\mathrm{id}_C, \circ(\sigma)]} [C, D].$$

We will use λ denote the composition of the maps from D_N to $[C, D]$ so we can write $\hat{P}$ as

$$\hat{P} \colon NT \xrightarrow{N} D_N \xrightarrow{\lambda} [C, D].$$

Now we refer to the diagram of Definition 0.1. Take the following substitutions for our variable names:

$$NT = NT_V(F,G),\ C = C(a,b),\ D = D(F(a), G(b))$$

We will take two different choices of substitutions for the remaining variables. First we take

$$\begin{aligned} &N = N_a, \quad D_N = D_{N_a} = D(F(a), G(a)), \\ &K = G, \quad D_K = D_G = D(G(a), G(b)), \\ &\sigma = \tau. \end{aligned}$$

This choice of variables gives us a map $\hat{P}_1 \colon NT \xrightarrow{\eta^C_{NT}} [C, NT \otimes C] \xrightarrow{[\mathrm{id}_C, P_1]} [C, D]$ where P_1 is the morphism P given above. The other substitutions we will make are

$$\begin{aligned} &N = N_b, \quad D_N = D_{N_b} = D(F(b), G(b)), \\ &K = F, \quad D_K = D_F = D(F(a), F(b)), \\ &\sigma = \mathrm{id}. \end{aligned}$$

Again this gives us a maps P_2 and $\hat{P}_2 \colon NT \xrightarrow{\eta^C_{NT}} [C, NT \otimes C] \xrightarrow{[\mathrm{id}_C, P_2]} [C, D]$. Definition 0.1 tells us that P_1 and P_2 are equal and so $\hat{P}_1 = P_1 \circ \eta^{C(a,b)}_{NT(F,G)} = P_2 \circ \eta^{C(a,b)}_{NT(F,G)} = \hat{P}_2$. By the working above each of $\hat{P}_1$ and $\hat{P}_2$ are given by compositions $N_a \circ \lambda_1$ and $N_b \circ \lambda_2$ respectively which exactly gives us the commuting diagram required. □

We can now prove Theorem 0.2

Proof. First we need to find maps α and β and e such that

$$\left(\prod_c N_c\right) \circ \alpha = \left(\prod_c N_c\right) \circ \beta.$$

We take α and β to be the products over objects in C of the maps λ_1 and λ_2 of Lemma 0.3 respectively. Lemma 0.3 gives us that $\prod_c N_c$ equalises α and β as required.

Finally we need to show that $NT(F, G)$ satisfies the universal property of this equaliser. If there exists an object M and for each $c \in C$ a morphism $M_c \colon M \to D(F(c), G(c))$ then Definition 0.1 gives us the existence of a unique map $\phi \colon M \to NT(F, G)$ such that $(\phi \otimes \mathrm{id}_{C(a,b)})$ factors through the diagram of Definition 0.1. In particular this means that ϕ is a unique map such that $M_a = N_a \circ \phi$ for all $a \in C$ and so ϕ is the unique map that factors through the diagram of Lemma 0.3 and by taking the product $\prod_c \phi$ factors uniquely through the equaliser and so satisfies the universal property as required. □

References

[1] Simon Willerton. "Enriched categories through examples". unpublished manuscript. Apr. 2022.